Coalition and Network Formation

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1 Coalitions and Networks in Economic Analysis

The formation of groups and networks is undoubtedly a central theme in the social sciences. Sociologists have long studied the formation of social groups and the importance of social networks, psychologists have discussed the importance of group behavior and group influence on individual behavior, political scientists have always been strongly interested in the formation of lobbies and political groups. In economics, the importance of groups has also long been recognized. Most economic activity is conducted by groups rather than individuals. Consumption is carried out by households instead of individuals, wage bargaining usually occurs among groups of workers and employers, economic decision are taken by groups of countries instead of individual states,... The list of groups participating in economic activities can be extended at will.

The economist’s approach to group and network formation is usually quite different from the approach of other social scientists. Economists value the importance of rationality and optimality, and the central question they pose is the following: How do self-interested agents decide to form groups and networks? The emphasis is thus put on the processes of group and network formation, and the computations that lead rational agents to choose to belong to groups and form links among themselves. In contrast, most other social sciences take the existence and groups and social networks as a given, and study how agents’ behavior is affected by their membership to some group and social network.

The formal analysis of group formation can be traced back to von Neumann and Morgenstern’s seminal book on game theory (“Theory of Games and Economic Behavior”) initially published in 1944. Starting with the study of two-player games, von Neumann and Morgenstern very soon discuss the extension of the theory to larger numbers of players, and emphasize the importance of the formation of groups (coalitions in the parlance of game theory) in the study of strategic situations. The issue of coalition formation has since been a central aspect of cooperative game theory, leading to the

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development of a number of cooperative solution concepts (core, bargaining sets...). In recent years, the study of coalition formation has been revived, due to the development of a number of applications in economics, and with a slight change on emphasis. The recent literature proposes to look at coalition formation as a non-cooperative process, by explicitly spelling out the procedures by which individual players form groups and networks.

Since the renewed interest in group formation is mainly motivated by economic applications, it is instructive to review some of the economic problems which require an analysis of coalition formation. In Industrial Organization, the formation of cartels and alliances has long been an object of study. In recent years, the development of new forms of competition, involving a mix of cooperation and competition among firms (for example, firms participating in joint research projects but competing on the marketplace) has stirred up a new interest in coalition formation. In International Economics, the formation of customs unions and free trade areas, has a distinguished history, but recent developments (the formation of new unions in North and South America – NAFTA and Mercosur, and the emergence of three trading blocs in Europe, America and the Pacific Basin) have led to a renewed interest in coalition formation. In Public Economics, the formation of local jurisdictions, and the provision of local public goods have recently gained a lot of attention, with the break up of some countries (U.S.S.R, Yugoslavia and Czechoslovakia), and the increasing regional tensions in a number of countries. In International Macroeconomics, the formation of monetary unions is clearly a hot topic of debate with the introduction of the euro. In Labor Economics, the formation of trade unions and the existence of different structures of trade unions in different countries has always been a puzzle. In environmental economics, the importance of transboundary pollution and the formation of groups of countries to negotiate international treaties of pollution abatement have clearly become central topics of discussion. Finally, the new political economy, investigating political institutions, has recently emphasized the importance of coalition formation in government cabinets and legislatures. In sum, the formation of coalitions is a pervasive phenomenon, which seems to permeate all areas of applied economics.

In order to understand the recent contributions to the theory of coalition and network formation, it is useful to distinguish three possible representations of gains from cooperation, in increasing degree of generality.

Coalitional Representation

In the coalitional representation, one associates to each subgroup $C$ of agents a monetary value $v(C)$ representing the total amount that the coalition can obtain by itself. This is interpreted as the worth of the coalition, which can be divided among its members.

Partition Function Representation
In the partition function representation, externalities across coalitions are taken into account. To each coalition structure \( \pi = \{C_1, \ldots, C_R\} \), one associates a vector of payments for all the coalitions in \( \pi \). \( v(C_r; \pi) \) then denotes the payment of coalition \( C_r \) when the coalition structure \( \pi \) is formed. This representation carries more information than the coalitional representation, because the payment of a coalition may depend on the way other coalitions are organized.

**Graph Representation**

In the graph representation, one is given the value \( v(g) \) of any graph \( g \) formed by the players. The literature distinguishes between two types of graph values. Component additive values assume that the value of a graph can be decomposed into the sum of values of its components. This implies that there are no externalities across components (the value of a component does not depend on the way other players are organized). Nonadditive values allow for externalities across components, just as partition functions allow for externalities across coalitions. Note that the graph representation is more general than the coalitional representation, because it conveys information about the way players are linked inside a component.

The three representations (coalitional, partition function and graph representations) have been considered in the literature, and analyzed using similar techniques. In applications, the use of one or the other representation is usually dictated by the structure of the economic model. In the remaining of the chapter, we abstract away from the issue of division of the payoffs inside a coalition and inside a component. We shall assume a fixed sharing rule, and let \( v^i(C), v^i(\pi) \) and \( v^i(g) \) denote the payoff of player \( i \) in coalition \( C \), in partition \( \pi \) and in graph \( g \) respectively. Alternatively, we can interpret the assumption of a fixed sharing rule as the non-transferability of payoff across agents in a coalition and in a graph.

## 2 Cooperative Solutions to Group and Network Formation

The earliest attempts to understand the formation of groups relied on cooperative solution concepts. We will review these concepts for the three representations outlined above. We will focus on solution concepts related to the core, which are the most prevalent concepts proposed in the literature. We note however that some papers have developed alternative concepts based on bargaining sets or the von Neumann and Morgenstern stable sets.

When one considers the coalitional representation, the core is easily defined. A coalition structure \( \pi = \{C_1, \ldots, C_R\} \) belongs to the core if and only
if there is no coalition of agents, \( S \), such that \( v^i(S) > v^i(C(i)) \forall i \in S \), where \( C(i) \) denotes the coalition \( i \) belongs to in the partition \( \pi \). Conceptually, it may be sometimes difficult to understand why coalitions form at all in the coalitional representation. If players have access to the same strategies individually and inside groups, there is no reason to believe that an extension of the group could reduce the payment of players. This argument has been used to justify the fact that coalitional games are superadditive, i.e. \( v(S \cup T) \geq v(S) + v(T) \) for any disjoint subsets \( S \) and \( T \). But if a game is superadditive, one should always expect the grand coalition to form, and the issue of coalition formation is irrelevant. This argument – showing that coalition formation is not an issue in the coalitional representation – has been challenged by a number of authors who point out that some external rigidities are present, leading to games which are not superadditive. For example, in jurisdiction and club formation, congestion might reduce the payoff of a coalition when too many individuals enter. Another example comes from rigidities in the political institutions. For example, if heterogeneous voters vote for a proportional tax rate in a jurisdiction to provide a public good, different voters having different preferences may benefit from seceding and forming smaller groups. Finally, if players have asymmetric information, the cost of forming large coalitions may increase, so that the game becomes non superadditive.

When one considers the partition representation, superadditivity is by no means guaranteed. For example, in an association of firms, accepting new members may reduce the competitive advantage of the standing members, and result in lower payoffs for them. Similarly, outsiders free-riding on the formation of a cartel or on the provision of a public good have no incentive to join a coalition and start contributing to the cartel or the public good. When one considers extending the core to games in partition function form, one immediately faces a conceptual problem. When a group of players deviates, it must predict the reaction of other players to the deviation. Four solution concepts have been proposed in the literature. A coalition structure \( \pi \) is core-stable if there does not exist a subset \( S \) of players and a partition \( \pi'_{N \setminus S} \) of the other players such that \( v^i(S, \pi'_{N \setminus S}) > v^i(\pi) \) for all players \( i \) in \( S \). In words, a coalition deviates whenever there exists a partition \( \pi' \) under which all the members of the coalition are better off. This specification assumes an extremely optimistic behavior on the part of members of the deviating coalition. As deviations are easy to engineer, core stable coalition structures are usually difficult to find. On the other extreme, one could consider the \( \alpha \) stable coalition structures. A coalition structure \( \pi \) is \( \alpha \) stable if there does not exist a subset of players \( S \) such that, for all partitions \( \pi'_{N \setminus S} \) of the other players, \( v^i(S, \pi'_{N \setminus S}) > v^i(\pi) \). This specification presumes a very pessimistic prediction of members of the deviating coalition. They assume
that other players will re-organize in the worst possible way. Clearly, under the \( \alpha \) stability concept, deviations are difficult to carry out, and it will be easier to find \( \alpha \) stable coalition structures. Two other intermediate solution concepts have been proposed. In the \( \gamma \) formulation, when a group of players deviate, all members of the coalition they break away. Hence, when a coalition \( S \) deviates, we need to keep track of the coalitions which were left by some members of \( S \). Let \( C_1, \ldots, C_S \) be those coalitions and \( C_{S+1}, \ldots, C_R \) the coalitions left intact. A coalition structure \( \pi \) is \( \gamma \) stable if there does not exist a coalition \( S \) such that for all \( i \) in \( S \), \( \nu^i(S, \{j \mid j \notin S, j \in C_i\}, C_{S+1}, \ldots, C_R) > \nu^i(\pi) \). Finally, in the \( \delta \) formulation, when a group deviates it supposes that members of the coalitions which lost some members stick together. Hence, a coalition structure \( \pi \) is \( \delta \) stable if there exists no coalition \( S \) such that, for all \( i \) in \( S \), \( \nu^i(S, C_1 \setminus \{j \mid j \notin S \cap C_1\}, \ldots, C_S \setminus \{j \in S \cap C_S\}, C_{S+1}, \ldots, C_R) > \nu^i(\pi) \). The four solution concepts (core stability, \( \alpha \), \( \gamma \) and \( \delta \) stability) cover the range of possible reactions of external players. As we will see below, they will lead to very different predictions in applications.

Concerning the graph representation, the first solution concept proposed by Jackson and Wolinsky in 1996 ("A Strategic Model of Social and Economic Network", *Journal of Economic Theory* 71, 44-74), is a local stability concept, based on an examination of a graph link per link. A graph is called pairwise stable if, whenever a link is formed both agents have an interest in forming the link \( (\nu^i(g \cup ij) \geq \nu^i(g) \) and \( \nu^j(g \cup ij) \geq \nu^j(g) \), and whenever a link is not formed, one of the agents has a strict incentive not to form it \( (\nu^i(g \cup ij) > \nu^i(g) \Rightarrow \nu^j(g \cup ij) < \nu^j(g)) \). This solution concept suffers from various shortcomings. By looking at a graph link per link, it does not recognize that the unit of decision in a graph is the agent, and not the pairwise links. A more demanding but more satisfactory solution concept has recently been proposed. A graph is called strongly stable if there exists no coalition of agents who, by re-arranging their links, could get a strictly higher payoff.

3 Noncooperative Models of Group and Networks

In recent years, the attention of game theorists has been focussed on noncooperative procedures of group and network formation. The earliest attempts have dealt with coalition formation. A first category of procedures are simultaneous games, where all agents simultaneously announce the groups they want to form.

In games with open membership, players cannot prevent other players from joining their group. These games have appeared in the 70’s to discuss
the formation of cartels. The more general formulation is the following address game. Let $M$ be a set of messages, with more messages than players. Each player announces a message $m^j$ in $M$, and coalitions are formed by all players who have announced the same message.

Games with exclusive membership are games where players choose the coalitions they wish to form and hence have the ability to exclude other players. The two most prominent games of exclusive membership are the $\gamma$ and $\delta$ games which were initially discussed (under a different name) by Van Neumann and Morgenstern. Players’ strategy spaces are the set of all coalitions to which they belong: $S^i = \{C \subset N, i \in C\}$. The outcome functions differ in the two games. In the $\gamma$ game, a coalition is formed only when all its members unanimously agree to form the coalition, i.e. $C = s_i \forall i \in C$. In the $\delta$ game, a coalition is formed even if some members choose not to join the coalition, $C = \{i | s_i = C\}$. This difference in outcome functions, as we will see below, generates a huge difference in the equilibria of the game.

The corresponding noncooperative process in graphs is the following linking game. Players’ strategies are the set of links that they may form (or a subset of the other players with which they want to form links). $S^i = \{C, C \subset N \setminus \{i\}\}$. A link is formed if and only if both players have announced their desire to form the link: $ij$ is formed if and only if $i \in s_j$ and $j \in s_i$.

It should be clear that all the noncooperative processes outlined above give rise to a large number of Nash equilibria, reflecting coordination failures among the players. In the exclusive membership coalition formation games, the situation where no coalition is formed is always an equilibrium: when other agents announce that they do not want to form a coalition, choosing to remain independent is always a best response. Similarly, in the linking game, the empty graph always emerges as an equilibrium: as long as the other players do not agree to form a link, it is a best response for every player to remain isolated. Various routes have been taken to alleviate these coordination problems. In exclusive membership coalition formation games, it has been suggested to look at strong Nash equilibria or coalition proof Nash equilibria of the noncooperative game. Notice that, a coalition can be sustained as a strong Nash equilibrium of the $\gamma$ (respectively $\delta$) game if and only if this coalition is $\gamma$ (respectively $\delta$) stable. In the linking game, one has similarly considered cooperative-based refinements, like equilibria which are immune to deviations by pairs of players, or Strong and coalition proof Nash equilibria.

As an alternative to cooperative-based refinements, a recent literature has considered sequential games of coalition formation, where the (generi-
(cally) unique Subgame Perfect Equilibrium of the game gives another way to select an equilibrium of the game. These procedures are based on extensions of Rubinstein’s alternating-offers bargaining game. Different procedures have been proposed. In one of them, players announce coalitions and the division of payoffs inside the coalition. All prospective members then respond to the offer. If they all agree, the coalition is formed, and players exit the game. If one of them reject, time passes and the player who rejected becomes the proposer in the next period. (This procedure is termed the infinite horizon unanimity game). Another procedure looks at what happens when, after a rejection, players are randomly chosen to make the next offer. The construction of similar sequential procedures of graph formation remains an open (and very complex) problem in the field.

4 Applications

In order to illustrate the use of the different models of coalition and graph formation, we consider two standard applications in Industrial Organization. The first application deals with the formation of collusive groups, and the second one with the formation of strategic alliances.

4.1 Collusive Groups

The formation of a cartel is a typical example of a situation of group formation with positive externalities. Players benefit from the formation of a cartel by the other players, since this entails an increase in market price. Formally, a game is said to exhibit positive externalities if whenever two coalitions merge, all external players are made better off. (Other examples of games with positive externalities include the provision of pure public goods.)

Consider a linear Cournot market, where market price is given by \( P = 1 - Q \) and firms have zero marginal cost. Let \( \pi \) denote the coalition structure formed by cartels on the market. A simple computation shows that each firm profit only depends on the total number of cartels formed, \( k \), and on the size of the cartel \( C(i) \) to which firm \( i \) belongs. Profit is given by:

\[
v^i(\pi) = \frac{1}{(k + 1)^2 |C(i)|}
\]
An interesting analysis is to consider how the function $v^i(\pi)$ varies when only one cartel of a varying size $k$ is formed on the market. The following picture graphs this function.

The solid line represents the profits of outsiders whereas the dashed line gives the profit of cartel members. It appears that the profit of cartel members is decreasing for small values of $k$ and then increasing. In particular, there is a unique value $k^*$ for which cartel members obtain the same payoff as if they were independent players ($k = 1$). This value is called the minimal profitable cartel size and it has been established that, in a linear Cournot market, this value amounts to roughly 80% of the firms in the industry.

The following table lists the equilibria of various games of coalition formation in this example. The proof of these statements is left as a (difficult) exercise to the reader.

<table>
<thead>
<tr>
<th>Game of coalition formation</th>
<th>Equilibrium coalition structures</th>
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</thead>
<tbody>
<tr>
<td>Open membership game</td>
<td>${1, 1, 1, \ldots, 1}$</td>
</tr>
<tr>
<td>$\gamma$ game</td>
<td>${k, 1, 1, \ldots, 1} \forall k \geq k^*$</td>
</tr>
<tr>
<td>$\delta$ game</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>Infinite horizon game</td>
<td>${k^*, 1, 1, \ldots, 1}$</td>
</tr>
</tbody>
</table>

To interpret the table, notice that, in a cartel of any size, each firm has an incentive to leave the cartel as long as all other cartel members stay put. Hence, both in the open membership game and in the $\delta$ game, no cartel of positive size can form. If, on the other hand, a departure triggers the complete dissolution of the cartel (as in the $\gamma$ game), any cartel of size greater than $k^*$ may emerge in equilibrium. The sequential procedure selects
one such cartel, where players form the minimal profitable coalition.

It is also possible to interpret collusion on the market as the formation of bilateral links between firms. A recent model has investigated what happens when firms can form market sharing agreements, by which they choose to stay out of each other’s market. Let $\pi(n)$ denote the profit that each firm makes on a market with $(n - 1)$ competitors, and define the total profit of a firm as

$$v^i(g) = \sum_{j \in i \notin g} \pi(n_j) + \pi(n_i).$$

In this model (as in the formation of cartels), firms benefit from the formation of links among other players (since this induces a reduction in the number of competitors on the market). It can be shown that pairwise stable graphs are characterized by the formation of complete components, of different sizes greater than a minimal threshold. In linear Cournot markets, two stable graphs emerge: the empty graph and the complete graph.

### 4.2 Strategic Alliances

In the second application, firms form groups in order to benefit from synergies in production, but remain competitors on the market. This is a model with negative externalities: when two groups of firms merge, all firms in the groups reduce their costs, and external firms obtain lower profits. (Other examples of games with negative externalities include the formation of customs unions, when firms can benefit from an increase in market size.)

We consider again a linear Cournot market, where inverse demand is given by $P = 1 - Q$. The constant marginal cost of each firm is a linearly decreasing function of the size of the alliance it belongs to, $c_i = \kappa - \lambda |A(i)|$. Direct computations then show that a firm’s profit is given by

$$v^i(\pi) = \frac{1 - \kappa}{n + 1} + \lambda |A(i)| - \frac{\lambda \sum |A(j)|^2}{n + 1}.$$ 

The following table lists, for four different games of coalition formation, the equilibrium coalition structures. (The proof is again left to the reader, to test her understanding of the various procedures of coalition formation.)

<table>
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<tr>
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<td>{N}</td>
</tr>
<tr>
<td>$\gamma$ game</td>
<td>$\emptyset$</td>
</tr>
<tr>
<td>$\delta$ game</td>
<td>{(3n + 1)/4, (n - 1)/4}</td>
</tr>
<tr>
<td>Infinite horizon game</td>
<td>{(3n + 1)/4, (n - 1)/4}</td>
</tr>
</tbody>
</table>
In this game, the only equilibrium of the open membership game is the grand coalition, as every player always has an incentive to join a group. On the other hand, if membership is exclusive, players have an incentive to form smaller subgroups, in order to benefit from cost asymmetries between firms. Typically, this will induce the formation of two groups of unequal sizes, where firms in the first group choose to increase their size in order to prevent the formation of a strong complementary group. The $\gamma$ game does not admit any strong Nash equilibrium in this case. The intuition for this result is somewhat difficult to grasp. Note that, if a group is larger than $n/2$, a subset of size $n/2$ has an incentive to deviate, knowing that the other firms will remain isolated. On the other hand the formation of two groups of size $n/2$ cannot be a strong Nash equilibrium, since any subset of players of size greater than $n/2$ would benefit from forming a group.

It should be noted that the formation of two asymmetric groups depends strongly on the fact that alliances are considered here as multilateral (rather than bilateral) agreements. If one considers an alternative model where firms derive cost synergies from the formation of pairwise links with other firms, the results are strikingly different. It can be shown that in that case, in a linear Cournot model, the only pairwise stable graph is the complete graph.