Elements of Viability Theory
for the Regulation of the Evolution of
the Architecture of Networks

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September 7, 2001

Abstract

The main purpose of viability theory is to explain the evolution of the state of a control system, governed by nondeterministic dynamics and subjected to viability constraints, to reveal the concealed feedbacks which allow the system to be regulated and provide selection mechanisms for implementing them. It assumes implicitly an “opportunistic” and “conservative” behavior of the system: a behavior which enables the system to keep viable solutions as long as its potential for exploration (or its lack of determinism) — described by the availability of several evolutions — makes possible its regulation.

Examples of viability constraints are provided by architectures of networks imposing constraints described by connectionist tensors operating on coalitions of actors linked by the network. The question raises how to modify a giving dynamical system governing the evolution of the signals, the connectionist tensors and the coalitions in such a way that the architecture remains viable.

1 Introduction

The purpose of this paper is to present a brief introduction of viability theory that can be relevant for the study of the regulation of the evolution

1. **cognitive systems** that must design learning processes allowing them to adapt and evolve in an environment defined by viability constraints in unexpected environmental circumstances, before reaching another subset regarded as a target if needed

2. of the **architecture of a network** described by connectionist tensors operating on coalitions of actors.

\(^1\)This paper is prepared for the Ecole Thématique du CNRS “Economie Cognitive”. The author thanks warmly Paul Bourgine for inviting this contribution.
Indeed, the mathematical viability theory has been motivated in the first place by the contingent evolution of the state of systems adapting to viability or scarcity constraints under uncertainty arising in biology, cognitive sciences and economics. These are the very same questions that are addressed in [1, 8, 6, Aubin], where one can find more details on viability theory and its applications to cognitive sciences (see [2, Aubin]) and dynamical economics (see [3, Aubin]).

On the other hand, neural networks, genetic networks and socio-economic networks describe collective phenomena through constraints relating actions of several actors, coalitions of these actors and multilinear connectionist operators acting on the set of actions of each coalition. The problem arises of correcting initial dynamics in such a way that the evolution of the architecture of such a network remains viable.

It is time to cross the interdisciplinary barrier and to confront and hopefully to merge the points of view rooted in different disciplines. Mathematics, thanks to its abstraction power by isolating only few key features of a class of problems, can help to bridge these barriers as long as it proposes new methods motivated by these new problems instead of applying the classical ones only motivated until now by physical sciences. Paradoxically, the very fact that the mathematical tools useful for social sciences are and have to be quite sophisticated impairs their acceptance by many social scientists and economists, and the gap menaces to widen.

1.1 The Constrained Set and the Target

The state variables of the systems under investigations range over the constrained subset $K$ — the environment — of the state space $X := \mathbb{R}^n$. One can also introduce a subset $C \subset K$ — the target — that can be taken empty if no objective is assigned to the system. We denote by $K\setminus C$ the subset of elements of the constrained set $K$ outside the target $C$.

An evolution $t \in [0,T] \mapsto x(t) \in X$ of the state of the system is said to be viable in $K$ on the interval $[0,T]$ before it reaches the target $C$ if

$$\forall t \in [0,T], \quad x(t) \in K\setminus C$$

The evolution is captured (in finite time $T$) if $x(T) \in C$. When $C := \emptyset$ and $T := +\infty$, we simply say that the evolution is viable in $K$. 
1.2 The Dynamical Behavior of an Animat

Next, we provide the mathematical description of the “engine” governing the evolution of the state. We assume that there exists a control parameter, or, better, a regulatory parameter, called a regulon, that influences the evolution of the state of the system. For instance, in the case of cognitive systems, this regulon can be regarded as a sensory-motor regulon that both perceives the state of the environment and acts on the cognitive system through a dynamical system (see chapter 8 of [2, Aubin] for more details on cognitive systems.) This dynamical system takes the form of a control system with (multivalued) feedbacks:

\[
\begin{align*}
\text{i)} & \quad x'(t) = f(x(t), u(t)) \quad \text{(action)} \\
\text{ii)} & \quad u(t) \in U(x(t)) \quad \text{(perception)}
\end{align*}
\]  

(1)

taking into account the a priori availability of several regulons \( u(t) \in U(x(t)) \) chosen in a subset \( U(x(t)) \subseteq Y \) of another finite dimensional vector-space \( Y \) subjected to state-dependent constraints.

Once the initial state is fixed, the first equation describes how the sensory-motor regulon acts on the velocities of the system whereas the second inclusion shows how the state (or an observation on the state) can be perceived through (several) sensory-motor regulons in a vicariant way.

We observe that there are many evolutions starting form a given initial state \( x_0 \), one for each time-dependent regulon \( t \mapsto u(t) \). The set-valued map \( U : X \rightharpoonup Y \) also describes the state-dependent constraints on the regulons. In this case, system (1) can no longer be regarded as a parametrized family of differential equations, as in the case when \( U(x) \equiv U \) does not depend upon the state, but as a differential inclusion (see [11, Aubin & Cellina] for example). Fortunately, differential inclusions enjoy most of the properties of differential equations.

An evolution of system (1) is a function \( t \mapsto x(t) \) satisfying this system for some (measurable) open-loop control \( t \mapsto u(t) \) almost everywhere.

For networks defined by connectionist tensors and coalitions of actors, the regulons are not given \textit{a priori} but built as ”viability multipliers” in such a way that the architecture of the network is viable.
1.3 The Simplest Example

We take $X = Y = \mathbb{R}$, $K := [a, b]$ where $0 < a < b$, $C = \emptyset$ and a system of the form

$$\begin{align*}
   i) & \quad x'(t) = u(t)x(t) \\
   ii) & \quad u(t) \in [-u, u] 
\end{align*}$$

where the regalon is simply chosen to be the growth rate of the system, only subjected to vary between lower and upper bounds. We shall pursue the study of this example along the way.

For other low-dimensional examples, see Chapter 2 of [3, Aubin].

2 Characterization of Viability and/or Capturability

The problems we shall study are all related to the viability of the constrained set and/or the capturability of a target under the dynamical system modeling the dynamic behavior of the cognitive system.

2.1 Definitions

The viability of a subset $K$ under a control system is a consistency property of the dynamics of the system confronted to the constraints it must obey during some length of time.

Namely, a subset $K \setminus C$ is locally viable under the control system described by $(f, U)$ if for every initial state $x_0 \in K$, there exist a positive time $T_{x_0} > 0$ and at least one solution to the system starting at $x_0$ and which is viable in $K$ on the interval $[0, T_{x_0}]$ before it reaches the target $C$ at time $T \geq T_{x_0}$. When $C := \emptyset$, we shall say that $K$ is viable if we can take $T_{x_0} := +\infty$ for all $x_0 \in K$.

To say that a singleton $\{c\}$ is viable amounts to saying that the state $c$ is an equilibrium — sometime called a fixed point. The trajectory of a periodic solution is also viable.

Contrary to the century-old tradition going back to Lyapunov, we require the system to capture the target $C$ in finite time, and not in an asymptotic way, as in mathematical models of physical systems. This is in particular mandatory for cognitive systems that must achieve tasks in finite time. However, there are close mathematical links between the various concepts of stability and viability. For instance, Lyapunov functions can be constructed
using tools of viability theory. This needs much more space to be described: we refer to Chapter 8 of [1, Aubin] and Chapter 8 of [3, Aubin] for more details on this topic.

The first task is to characterize the subsets having this viability/capturability property. To be of value, this task must be done without solving the system for checking the existence of viable solutions for each initial state.

An immediate intuitive idea jumps to the mind: at each point on the boundary of the constrained set outside the target, where the viability of the system is at stake, there should exist a velocity which is in some sense tangent to the viability domain and serves to allow the solution to bounce back and remain inside it. This is, in essence, what the viability theorem below states. Before stating it, the mathematical implementation of the concept of tangency must be made.

We cannot be content with viability sets that are smooth manifolds (such as spheres, which have no interior), because inequality constraints would thereby be ruled out (as for balls, that possess distinct boundaries). So, we need to “implement” the concept of a direction $v$ tangent to $K$ at $x \in K$, which should mean that starting from $x$ in the direction $v$, we do not go too far from $K$: The adequate definition due to G. Bouligand and F. Severi proposed in 1930 states that a direction $v$ is tangent to $K$ at $x \in K$ if it is a limit of a sequence of directions $v_n$ such that $x + h_n v_n$ belongs to $K$ for some sequence $h_n \to 0^+$. The collection of such directions, which are in some sense “inward”, constitutes a closed cone $T_K(x)$, called the tangent cone\(^2\) to $K$ at $x$. Naturally, except if $K$ is a smooth manifold, we lose the fact that the set of tangent vectors is a vector-space, but this discomfort in not unbearable, since advances in set-valued analysis built a calculus of these cones allowing us to compute them. See [14, Aubin & Frankowska] and [30, Rockafellar & Wets] for instance.

### 2.2 The Adaptive Map

We then associate with the dynamical system (described by $(f, U)$ and with the viability constraints (described by $K$) the (set-valued) adaptive or regulation map $R_K$. It maps any state $x \in K \setminus C$ to the subset $R_K(x)$ (possibly empty) consisting of sensory-motor regulons $u \in U(x)$ which are viable in

\(^2\)replacing the linear structure underlying the use of tangent spaces by the tangent cone is at the root of Set-Valued Analysis.
the sense that \( f(x, u) \) is tangent to \( K \) at \( x \):

\[
R_K(x) := \{ u \in U(x) \mid f(x, u) \in T_K(x) \}
\]

We can for instance compute the adaptive map in many instances.

**Example:** The regulation map is equal to

\[
R_K(x) := \begin{cases} 
[0, +\infty] & \text{if } x = a \\
[-u_0, +\infty] & \text{if } x \in [a, b[ \\
[-u_0, 0] & \text{if } x = b 
\end{cases}
\]

It is set-valued, has nonempty values, but has too poor continuity property, a source of mathematical difficulties, as we shall see.

### 2.3 The Viability Theorem

The Viability Theorem states that *the target \( C \) can be reached in finite time from each initial condition \( x \in K \setminus C \) by at least one evolution of the control system viable in \( K \) if and only if for every \( x \in K \setminus C \), there exists at least one viable control \( u \in R_K(x) \).

This Viability Theorem holds true when both \( C \) and \( K \) are closed and for a rather large class of systems, called Marchaud systems: Beyond imposing some weak technical conditions, the only severe restriction is that, for each state \( x \), the set of velocities \( f(x, u) \) when \( u \) ranges over \( U(x) \) is convex\(^3\).

Curiously enough, when the constrained set is assumed to be viable, convex and compact, then one can prove that there exists a (viable) equilibrium. Without convexity, we deduce only the existence of minimal viable closed subsets.

The proofs of the above Viability Theorem and the Equilibrium Theorem are difficult, but their consequences are much easier to obtain and can be handled with moderate mathematical competence.

**Example:** The interval \([a, b]\) is viable under system (2).

\(^3\)This happens for the class of control systems of the form

\[
x'(t) = f(x(t)) + G(x(t))u(t)
\]

where \( G(x) \) are linear operators from the control space to the state space, when the maps \( f : X \rightarrow X \) and \( G : X \rightarrow \mathcal{L}(Y, X) \) are continuous and when the control set \( U \) (or the images \( U(x) \)) are convex.
2.4 The Adaptation Law

Once this is done, and whenever a constrained subset is viable for a cognitive system, the second task is to show how to govern the evolution of viable evolutions. We thus prove that an evolution of system (1) is viable if and only if it is governed by

\[
\begin{aligned}
&i) \quad x'(t) = f(x(t), u(t)) \\
&ii) \quad u(t) \in R_K(x(t)) \quad \text{(adaptation law)}
\end{aligned}
\]

until the state reaches the target \( C \).

We observe that the initial set-valued map \( U \) involved in (1)(ii) is replaced by the adaptive map \( R_K \) in (3)(ii). The inclusion \( u(t) \in R_K(x(t)) \) can be regarded as an adaptation law (rather than a learning law, since there is no storage of information at this stage of modeling).

2.5 Planning Tasks: Qualitative Dynamics

Reaching a target is not enough for studying the behavior of cognitive systems, that have to plan tasks in a given order. This issue has been recently revisited in [13, Aubin & Dordan] in the framework of qualitative physics (see [22, Dordan] and [2, Aubin] for more details on this topic). We describe the sequence of tasks or objectives by a family of subsets regarded as qualitative cells. Given an order of visit of these cells, the problem is to find an evolution visiting these cells in the prescribed order.

2.6 The Metasystem

In order to bound the chattering (rapid oscillations or discontinuities) of the regulons, we set a priori constraints on the velocities of the form

\[
\forall t \geq 0, \quad u'(t) \in \Phi
\]

where \( \Phi \subset Y \) is a bounded subset of the space of regulons (that may depend on both the state and the regulon). The metasystem associated with the initial viability problem is the system

\[
\begin{aligned}
&i) \quad x'(t) = f(x(t), u(t)) \\
&ii) \quad u'(t) \in \Phi
\end{aligned}
\]

subjected to the metaconstraints

\[
\forall t \geq 0, \quad x(t) \in K & \quad u(t) \in U(x(t))
\]
Unfortunately, the above metaconstraints may no longer be viable under the metasystem.

**Example:** We require that the velocity of the regulon (equal here to the growth rate) must remain between bounds $-d$ and $+c$. Therefore, the metasystem is

$$\begin{align*}
&i) \quad x'(t) = u(t)x(t) \\
&ii) \quad u'(t) \in \Phi := [-d, +c] \\
\end{align*}$$

subjected to metaconstraints

$$\forall \ t \geq 0, \ (x(t), u(t)) \in [a, b] \times [-\bar{u}, +\bar{u}]$$

One can prove that these metaconstraints are not viable under the metasystem.

### 2.7 Restoring Viability

The above example shows that there are no reasons why an arbitrary subset $K$ should be viable under a control system. Therefore, the problem of reestablishing viability arises. One can imagine several methods for this purpose:

1. Keep the constraints and change initial dynamics by introducing regulons that are "viability multipliers"

2. Keep the same dynamics and looking for viable constrained subsets

3. Change either the dynamics or the set of constraints

   (a) either by changing the regulons according to feedbacks or dynamic feedbacks that can be constructed (see for instance [1, 3, Aubin]),

   (b) or by letting the set of constraints evolve according to mutational equations, as in [5, Aubin].

4. or change the initial conditions by introducing a reset map $R$ mapping any state of $K$ to a (possibly empty) set $R(x) \subset X$ of new "initialized states" (impulse control).

We shall describe succinctly these methods.
3 Viability Kernels and Capture basins

When a closed subset $K$ is not viable under a control system, then two questions arise naturally: find solutions starting from $K$ which remain viable in $K$ as long as possible, and starting outside of $K$, find solutions which return to $K$ as soon as possible to restore the viability. These natural questions justify the introduction of the following concepts:

1. The subset $\text{Viab}(K)$ of initial states $x_0 \in K$ such that one solution $x(\cdot)$ to system (1)ii) starting at $x_0$ is viable in $K$ for all $t \geq 0$ is called the viability kernel of $K$ under the control system. A subset $K$ is a repeller if its viability kernel is empty.

2. The subset $\text{Capt}^K(C)$ of initial states $x_0 \in K$ such that the target $C \subset K$ is reached in finite time before possibly leaving $K$ by one solution $x(\cdot)$ to system (1)ii) starting at $x_0$ is called the viable-capture basin of $C$ in $K$. A subset $C \subset K$ such that $\text{Capt}^K(C) = C$ is said to be isolated in $K$.

One can prove that if the system is Marchaud and if $K$ is closed, the viability kernel $\text{Viab}(K)$ of the subset $K$ is the largest closed subset of $K$ viable under the control system. Hence, all interesting features such as equilibria, trajectories of periodic solutions, limit sets and attractors, if any, are all contained in the viability kernel.

**Example:** Viability kernel of metaconstraints Since the metaconstraints (7) are not viable metasystem (6), one can compute the viability kernel of $[a, b] \times [-\bar{u}, +\bar{u}]$. It is the subset of pairs $(x, u)$ such that

$$u \in G_K(x) := \left[ -2c\log\left(\frac{x}{a}\right) + \sqrt{2d\log\left(\frac{b}{x}\right)} \right] \quad \square$$

(8)

One can also prove that the viability kernel is the unique closed subset $D \subset K$ viable and isolated in $K$ such that $K \setminus D$ is a repeller. If $K$ is a repeller and $C \subset K$ is closed, one can prove that the capture basin $\text{Capt}^K(C)$ of $C \subset K$ is the unique closed subset $D$ between $C$ and $D$ such that $D$ is isolated in $K$ and $D \setminus C$ is locally viable.

The viability kernels of a subset and the capture basins of a target can thus be characterized in diverse ways through tangential conditions thanks to the viability theorems. They play a crucial role in viability theory, since many interesting concepts are often viability kernels or capture basins.
Furthermore, algorithms designed in [31, Saint-Pierre] allow us to compute viability kernels and capture basins (see also [20, Cardaliaguet, Quincampoix & Saint-Pierre] and [29, Quincampoix & Saint-Pierre]). In general, there are no explicit formulas providing the viability kernel and capture basins.

**Remark: The Crisis Function** — If the solution is allowed to leave the constrained set $K$, one can use the crisis function introduced in [23, Doyen & Saint-Pierre], which measures the minimal time spent by the evolutions $x(\cdot)$ outside $K$.

4 Selecting Viable Feedbacks

4.1 Static Feedbacks

A (static) feedback $r$ is a map $x \in K \mapsto r(x) \in X$ which is used to pilot evolutions governed by the differential equation $x'(t) = f(x(t), r(x(t)))$. A feedback $r$ is said to be viable if the solutions to the differential equation $x' = f(x, r(x))$ are viable in $K$. The most celebrated examples of linear feedbacks in linear control theory designed to control a system have no reason to be viable for an arbitrary constrained set $K$, and, according to the constrained set $K$, the viable feedbacks are not necessarily linear.

However, the Viability Theorem implies that a feedback $r$ is viable if and only if $r$ is a selection of the adaptive map $R_K$ in the sense that

$$\forall x \in K \backslash C, \quad r(x) \in R_K(x)$$

Hence, the method for designing feedbacks for cognitive systems to evolve in a constrained subset amounts to find selections $r(x)$. One can design a “factory” for designing selections (see Chapter 6 of [1, Aubin], for instance). Ideally, a feedback should be continuous to guarantee the existence of a solution to the differential equation $x' = f(x, r(x))$. But this is not always possible. This is the case of slow selection $r^o$ of $R_K$ of minimal norm, governing the evolution of slow viable evolutions (despite its lack of continuity).

**Example:** The slow selection $r^o$ for the system (2) is equal to $r^o(a) = 0$ and $r^o(x) = -\hat{u}$ when $a < x \leq b$. It is discontinuous at $x = a$. 
4.2 Dynamic Feedbacks

One can also look for dynamic feedbacks $g_K : X : K \times Y \mapsto Y$ that governs the evolution of both the states and the regulons through the metasystem of differential equations

$$
\begin{align*}
\text{i)} & \quad x'(t) = f(x(t), u(t)) \\
\text{ii)} & \quad u'(t) = g_K(x(t), u(t))
\end{align*}
$$

(10)

A dynamic feedback $g_k$ is viable if the metaconstraints (5) are viable under the metasystem (10).

As for the (static) feedbacks, one can prove that all the viable dynamic feedbacks are selections of a dynamical adaptive map $G_K : K \times Y \leadsto Y$ obtained by “differentiating” the adaptation law (1)ii) thanks to the differential calculus of set-valued maps see [14, Aubin & Frankowska]).

4.3 Heavy Evolutions and the Inertia Principle

Among the viable dynamic feedbacks, one can choose the heavy viable dynamic feedback $g^*_K \in G_K$ with minimal norm that governs the evolution of heavy viable solutions, i.e., viable evolutions with minimal velocity. They are called “heavy” viable evolutions\(^4\) in the sense of heavy trends in economics.

Heavy viable evolutions offer convincing metaphors of the evolution of biological, economic, social and cognitive systems that obey the inertia principle. It states in essence that “the regulons are kept constant as long as viability of the system is not at stake”. Heavy viable evolutions can be viewed as providing mathematical metaphors for the concept of punctuated equilibrium introduced in paleontology by Eldredge and Gould in 1972. In our opinion, this is a mode of regulation of cognitive systems (see Chapter 8 of [2, Aubin] for further justifications).

Indeed, as long as the state of the system lies in the interior of the constrained set (i.e., away of its boundary), any regulon will do. Therefore, the system can maintain the regulon inherited from the past. This happens if the system obeys the inertia principle. Since the state of the system evolves while the regulon remains constant, it may reach the viability boundary with an “outward” velocity. This event corresponds to a period of viability crisis: To survive, the system must find other regulons such that the new

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\(^4\)When the regulons are the velocities, heavy solutions are the ones with minimal acceleration, i.e., maximal inertia.
associated velocity forces the solution back inside the viability set until the
time when a regulon can remain constant for some time.

**Example:** One can prove that the set-valued map $G_K$ for the system
(2) is given by the formula (8). Hence the heavy dynamic feedback $g^H_K$ is
given by the formula

$$g^H_K(x, u) := \min \left( 0, + \sqrt{2d \log \left( \frac{b}{x} \right)} \right)$$

if $u > 0$, $g^H_K(x, 0) = 0$ and, if $u < 0$,

$$g^H_K(x, u) := \max \left( 0, - \sqrt{2c \log \left( \frac{x}{a} \right)} \right)$$

Starting from $(x_0, u_0)$ where $u_0 \in \left[ 0, \sqrt{2d \log \left( \frac{b}{x_0} \right)} \right]$, the heavy solution
$x(t) = x_0 e^{u_0 t}$ is associated with the constant growth rate $u(t) = u_0$ until the
time $t_1 = \frac{\log \left( \frac{b}{x_0} \right)}{u_0} - \frac{u_0}{2d}$ when $x(t_1) = be^{\frac{-u_0^2}{2d}}$ because $g^H_K(x(t_1)) = u_0$. Then
*this is the last moment when we have to change the regulon* by taking $u(t) =
\max(u_0 - d(t - t_1))$ that decreases with minimal velocity until the time $t_2 = t_1 + \frac{u_0}{d}$
when $x(t) = be^{\frac{-u(t)^2}{2d}}$ reaches the equilibrium $x(t_2) = b$ where $u(t_2) = 0$.

One can prove that all solutions $(x(t), u(t))$ of the metasystem starting from
$(x(t_1), u_0)$ at time $t_1$ either range the boundary of the viability kernel until
$t_2$ (this is the case when $u(t) = u_0 - d(t - t_1)$) or leave $[a, b] \times [-\infty, +\infty]$ in finite time. This illustrates a general property of the boundary of the
viability kernel, the barrier or semipermeability property discovered by Marc
Quincampoix: no solution starting from the boundary of the viability kernel
can enter its interior. One of them at least remains on the boundary, other
evolutions may leave the constrained set in finite time.

### 4.4 Warning Signal

*Imposing a speed limit on regulons conceals a warning signal to regulons,*
which must start to evolve as soon as the boundary of the viability kernel
is reached.

But one can design other ways to implement warning signals for avoiding
an impulsive change of regulon when the state is about to leave the con-
strained set $K$. A solution can be provided by replacing the constrained set
$K$ by a fuzzy set $\gamma_K : K \rightarrow [0, 1]$, that is a membership function equal to 0 outside $K$, the membership values $\gamma_K(x)$ ranging between 0 and 1 inside $K$. The case when $\chi_K(x) = 1$ whenever $x \in K$ provides the original (usual) set $K$. A standard way for building fuzzy sets is to take for membership function
\[
\delta_K(x) := \min \left( 1, \varepsilon \inf_{y \in K} \| x - y \|^2 \right)
\]
(see [12, Aubin & Dordan] for instance). The fuzzy set $\gamma_K$ being given, one can replace the original viability constraints by
\[
\forall t \geq 0, \quad \gamma_K(x(t)) \geq \gamma_K(x_0) e^{-\alpha t}
\]
where $\alpha > 0$. Assuming that the membership function $\gamma_K$ is differentiable for simplicity, we introduce the fuzzy regulation map $R_{\gamma_K}$ defined by
\[
R_{\gamma_K}(x) := \{ u \in U(x) \mid \langle \gamma'_K(x), f(x,u) \rangle + \alpha \gamma_K(x) \geq 0 \}
\]
We do not need to assume the membership function to be differentiable. In the case of the function $\delta_K$, one can prove that whenever $\delta_K(x) < 1$,
\[
R_{\delta_K}(x) := \{ u \in U(x) \mid \langle x - \pi x, f(x,u) \rangle + 2\varepsilon \alpha \delta_K(x) \geq 0 \}
\]
where $\pi$ is the projection onto the complement of $K$.

Then one can prove that fuzzy constrained set $\gamma_K$ is viable if and only if for any $x \in K$, $R_{\gamma_K}(x)$ is not empty and that the fuzzy viable evolutions are regulated by the regulation law
\[
\forall t \geq 0, \quad u(t) \in R_{\gamma_K}(x(t))
\]
If the above necessary and sufficient condition is not satisfied, one can also prove the existence of a fuzzy viability kernel.

One can also assume that the set-valued map $U$ is actually fuzzy, in the sense that it maps every $x$ to a fuzzy set of controls (see [12, Aubin & Dordan]).

### 4.5 Impulse Systems

There are many other dynamics that obey the inertia principle, among which heavy viable evolutions are the smoothest ones. At the other extreme, one can study also the (discontinuous) “impulsive” variations of the sensory-motor regulon. Instead of waiting the system to find a sensory-motor regulon that
remains constant for some length of time, as in the case of heavy solutions, one can introduce another (static) system that resets a new constant sensory-motor whenever the viability is at stakes, in such a way that the system evolves until the next time when the viability is again at stakes.

This regulation mode is a particular case of what are called impulse control in control theory (see for instance [6, Aubin], [17, Aubin, Lygeros, Quincampoix, Sastry & Seube]), hybrid systems in computer sciences (see for instance [32, Shaft & Schumacher]) and Integrate and Fire models in neuro-biology (see [19, Bressloff & Coombes] and [33, 34, Shimokawa, Pakdaman & Sato]), etc. Impulse systems are described by a control system governing the continuous evolution of the state between two impulsions, and a reset map resetting new initial conditions whenever the state enters the domain of the reset map.

An evolution governed by an impulse dynamical system, called a run in the control literature, is defined by a sequence of cadences (periods between two consecutive impulse times), of reinitialized states and of motives describing the “continuous” evolution along a given cadence, the value of a motive at the end of a cadence being reset as the next reinitialized state of the next cadence.

Given an impulse system, one can characterize the map providing both the next cadence and the next reinitialized state without computing the impulse system, as a set-valued solution of a system of partial differential inclusions. It provides a “summary” of the behavior of the impulse system from which one can then reconstitute the evolutions of the continuous part of the run by solving the motives of the run that are the solutions to the dynamical system starting at a given reinitialized state.

A cadenced run is defined by constant cadence, initial state and motive, where the value at the end of the cadence is reset at the same reinitialized state. It plays the role of “discontinuous” periodic solutions of a control system.

We prove in [15, Aubin & Haddad] that if the sequence of reinitialized states of a run converges to some state, then the run converges to a cadenced run starting from this state, and that, under convexity assumptions, that a cadenced run does exist.
4.6 Mutational Equations Governing the Evolution of the Constrained Sets

Alternatively, if the viability constraints can evolve, another way to resolve a viability crisis is to relax the constraints so that the state of the system remains inside the new viability set. For that purpose, kind of differential equation governing the evolution of subsets, called mutational equations, have been designed. This requires an adequate definition of the velocity $\dot{K}(t)$ of a “tube” $t \mapsto K(t)$, called mutation, that makes sense and allows us to prove results analogous to the ones obtained in the domain of differential equations. This can be done, but cannot be described in few lines.

Hence the viability problem amounts to find evolutions of both the state $x(t)$ and the subset $K(t)$ to the system

$$
\begin{align*}
\text{(i)} & \quad x'(t) = f(x(t), K(t)) \quad \text{(differential equation)} \\
\text{(ii)} & \quad \dot{K}(t) \ni m(x(t), K(t)) \quad \text{(mutational equation)}
\end{align*}
$$

(11)

viable in the sense that for every $t$, $x(t) \in K(t)$. For more details, see [5, Aubin].

5 Designing Regulons

When an arbitrary subset is not viable under a “intrinsic” system $x'(t) = f(x(t))$, the question arises to modify the dynamics by introducing regulons and designing feedbacks so that the constrained subset $K$ becomes viable under the new system. Using the above results and characterizations, one can design several mechanisms. We just describe three of them, that are described in more details in [3, Aubin]:

5.1 Viability Multipliers

If the constrained set $K$ is of the form

$$
K := \{x \in X \text{ such that } h(x) \in M\}
$$

where $h : X \to Z := \mathbb{R}^m$ and $M \subseteq Z$, we regard elements $u \in Z$ as viability multipliers, since they play a role analogous to Lagrange multipliers in optimization under constraints. They are candidates to the role of regulons
regulating such constraints. Indeed, we can prove that $K$ is viable under
the control system

$$x'_j(t) = f_j(x(t)) + \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} u_k(t)$$

in the same way than the minimization of a function $x \mapsto J(x)$ over a
constrained set $K$ is equivalent to the minimization without constraints of
the function

$$x \mapsto J(x) + \sum_{k=1}^{m} \frac{\partial h_k(x)}{\partial x_j} u_k$$

for an adequate Lagrange multiplier $u \in Z$.

### 5.2 Connection Matrices

Instead of introducing viability multipliers, we can use a connection matrix
$W \in \mathcal{L}(X, X)$ as in neural networks (see [2, Aubin] for instance). We replace
the intrinsic system $x' = f(x)$ (where $I$ denotes the identity) by the system

$$x'(t) = W(t) f(x(t))$$

and choose the connection matrices $W(t)$ in such a way that the solutions
of the above system are viable in $K$.

The evolution of the state no longer derives from intrinsic dynamical laws
valid in the absence of constraints, but requires some “self-organization” —
described by connection matrices — that evolves together with the state of
the system in order to adapt to the viability constraints, the velocity of connection matrices describing the concept of “emergence”. The evolution law
of both the state and the connection matrix results from the confrontation
of the intrinsic dynamics to the viability constraints.

One can prove that the regulation by viability multipliers $u$ is a particular
case of the regulation by connection matrices $W$: We associate with $x$ and
$u$ the matrix $W$ the entries of which are equal to

$$w_{i,j} := - \frac{f_i(x)}{\|f(x)\|^2} \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_j} u_k(t) \text{ if } i \neq j$$

and, of $j = i$,

$$w_{i,i} := 1 - \frac{f_i(x)}{\|f(x)\|^2} \sum_{k=1}^{m} \frac{\partial h_k(x(t))}{\partial x_i} u_k(t)$$
The converse is false in general. However, if we introduce the connectionist complexity index $\|I - W\|$, one can prove that the viable evolutions governed with connection matrices minimizing at each instant the connectionist complexity index are actually governed by the viability multipliers with minimal norms.

The concept of heavy evolution when the regulon is a connection matrix amounts to minimize the norm of the velocity $W'(t)$ of the connection matrix $W(t)$ starting from the identity matrix, that can be used as measure of dynamical connectionist complexity. Such a velocity could encapsulate the concept of “emergence” in the systems theory literature. The connection matrix remains constant — without emergence — as long as the viability of the system is not at stakes, and evolves as slowly as possible otherwise.

5.3 Hierarchical Organization

One can also design dynamic feedbacks for obeying constraints of the form

$$\forall t \geq 0, \ W^{m-1}(t) \cdots W^j(t) \cdots W^0(t)x(t) \in M \subset Y$$

is satisfied at each instant. Such constraints can be regarded as describing a sequence of $m$ planning procedures.

Introducing at each “level of such a hierarchical organization” $x_i(t) := W^i(t)x_{i-1}(t)$, one can design dynamical systems modifying the evolution of the intermediate states $x_i(t)$ governed

$$x_j^i(t) = g_j(x_j(t))$$

and the entries of the matrices $W^i(t)$ by

$$w_{k,l}^{j^i}(t) = e_{k,l}^j(W^j(t))$$

Using viability multipliers, one can prove that dynamical systems of the form

$$\begin{cases} 
(1)^0 & x_0^j(t) = g_0(x_0(t)) - W^0p_1(t)(j = 0) \\
(1)^j & x_j^i(t) = g_j(x_j(t)) + p_j(t) - W^jp_{j+1}(t) \\
& (j = 1, \ldots, m - 1) \\
(1)^m & x_m^i(t) = g_m(x_m(t)) + p_m(t) (j = m) \\
(2)^{j}_{(k,l)} & w_{k,l}^{j^i}(t) = e_{k,l}^j(W^j(t)) - x_{j_k}(t)p_{(j+1)_l}(t) \\
& (j = 0, \ldots, m - 1, \ k = 1, \ldots, n, \ l = 1, \ldots m) 
\end{cases}$$
govern viable solutions. Here, the viability multipliers $p_j$ are used as messages to both modify the dynamics of the $j$th level state $x_j(t)$ and to link to consecutive levels $j + 1$ and $j$.

Furthermore, the connection matrices evolve in a Hebbian way, since the correction of the velocity $W_j^l k$ of the entry is the product of the $k$th component of the $j$-level intermediate state $x_j$ and the $l$th component of the $(j + 1)$-level viability multiplier $p_{j + 1}$.

### 5.4 Evolution of the Architecture of a Network

This hierarchical organization is a particular case of a network. Indeed, the simplest general form of coordination is to require that a relation between actions of the form $g(A(x_1, \ldots, x_n)) \in M$ must be satisfied. Here $A : \prod_{i=1}^{n} X_i \rightarrow Y$ is a connectionist operator relating the individual actions in a collective way. Here $M \subset Y$ is the subset of the resource space $Y$ and $g$ is a map, regarded as a propagation map.

We shall study this coordination problem in a dynamic environment, by allowing actions $x(t)$ and connectionist operators $A(t)$ to evolve according to dynamical systems we shall construct later. In this case, the coordination problem takes the form

$$\forall t \geq 0, \ g(A(t)(x_1(t), \ldots, x_n(t))) \in M$$

However, in the fields of motivation under investigation, the number $n$ of variables may be very large. Even though the connectionist operators $A(t)$ defining the “architecture” of the network are allowed to operate a priori on all variables $x_i(t)$, they actually operate at each instant $t$ on a coalition $S(t) \subset N := \{1, \ldots, n\}$ of such variables, varying naturally with time according to the nature of the coordination problem.

Therefore, our coordination problem in a dynamic environment involves the evolution

1. of actions $x(t) := (x_1(t), \ldots, x_n(t)) \in \prod_{i=1}^{n} X_i$;

2. of connectionist operators $A_{S(t)}(t) : \prod_{i=1}^{n} X_i \rightarrow Y$;

3. acting on coalitions $S(t) \subset N := \{1, \ldots, n\}$ of the $n$ actors

and requires that

$$\forall t \geq 0, \ g \left( \{ A_{S(t)}(x(t)) \}_{S \subset N} \right) \in M$$
where \( g : \prod_{S \subseteq N} Y_S \rightarrow Y \).

The question we raise is the following. Assume that we may know the intrinsic laws of evolution of the variables \( x_i \) (independently of the constraints), of the connectionist operator \( A_S(t) \) and of the coalitions \( S(t) \), there is no reason why collective constraints defining the above architecture are viable under these dynamics, i.e, satisfied at each instant.

One may be able, with a lot of ingenuity and the intimate knowledge of a given problem, and for “simple constraints”, to derive dynamics under which the constraints are viable.

However, we can investigate whether there is a kind of mathematical factory providing classes of dynamics “correcting” the initial (intrinsic) ones in such a way that the viability of the constraints is guaranteed. One way to achieve this aim is to use the concept of “viability multipliers” \( q(t) \) ranging over the dual \( Y^* \) of the resource space \( Y \) that can be used as “controls” involved for modifying the initial dynamics.

This may allow us to provide an explanation of the formation and the evolution of the architecture of the network and of the active coalitions as well as the evolution of the actions themselves.

In order to tackle mathematically this problem, we shall

1. restrict the connectionist operators to be multiaffine, and thus, involve tensor products,

2. next, allow coalitions \( S \) to become fuzzy coalitions so that they can evolve continuously.

Fuzzy coalitions \( \chi = (\chi_1, \ldots, \chi_n) \) are defined by memberships \( \chi_i \in [0,1] \) between 0 and 1, instead of being equal to either 0 or 1 as in the case of usual coalitions. The membership \( \gamma_S(\chi) := \prod_{i \in S} \chi_i \) is by definition the product of the memberships of the members \( i \in S \) of the coalitions. Using fuzzy coalitions allows us to defined their velocities and study their evolution.

The viability multipliers \( q(t) \in Y^* \) can be regarded as regulons, i.e., regulation controls or parameters, or virtual prices in the language of economists. They are chosen adequately at each instant in order that the viability constraints describing the network can be satisfied at each instant, and the main theorem of this paper guarantees that it is possible. Another one tells us how to choose at each instant such regulons (the regulation law).

For each actor \( i \), the velocities \( x_i(t) \) of the state and the velocities \( \chi_i(t) \) of its membership in the fuzzy coalition \( \chi(t) \) are corrected by subtracting
1. the sum over all coalitions $S$ to which he belongs of adequate functions weighted by the membership $\gamma_S(\chi(t))$:

2. the sum over all coalitions $S$ to which he belongs of the costs of the constraints associated with connectionist tensor $A_S$ of the coalition $S$ weighted by the membership $\gamma_S \chi(t))$. This type of dynamics describes a panurigean effect. The (algebraic) increase of actor $i$'s membership in the fuzzy coalition aggregates over all coalitions to which he belongs the cost of their constraints weighted by the products of memberships of the actors of the coalition other than him.

As for the correction of the velocities of the connectionist tensors $A_S$, their correction is a weighted “multi-Hebbian” rule: for each component of the connectionist tensor, the correction term is the product of the membership $\gamma_S(\chi(t))$ of the coalition $S$, of the components $x_{ik}(t)$ and of the component $q^j(t)$ of the regulon.

In other words, the viability multipliers appear in the regulation of the multiaffine connectionist operators under the form of tensor products, implementing the Hebbian rule for affine constraints (see [2, ?, ?, Aubin]), and “multi-Hebbian” rules for the multiaffine ones, as in [10, Aubin & Burnod].

Even though viability multipliers do not provide all the dynamics under which a constrained set is viable, they provide classes of them exhibiting interesting structures that deserve to be investigated and tested in concrete situations.

To be more precise, we illustrate the above considerations in the case of only two actors: we assume now that $X := X_1 \times X_2$ is the product of two vector spaces. Affine constraints takes the form

$$\forall \, t \geq 0, \quad W_1(t)x_1(t) + W_2(t)x_2(t) + W_0(t) \in M$$

where $W_i \in \mathcal{L}(X_i, Y)$ ($i = 1, 2$) and $W_0 \in Y$. But we can also involve a bilinear operator $W_{(1,2)} \in \mathcal{L}_2(X_1 \times X_2, Y)$ and consider bi-affine constraints of the form:

$$\forall \, t \geq 0, \quad W_{(1,2)}(t)(x_1(t), x_2(t)) + W_1(t)x_1(t) + W_2(t)x_2(t) + W_0(t) \in M$$

We introduce the linear operators $W_{(1,2)}(x_i) \in \mathcal{L}(X_i, Y)$ defined by

$$W_{(1,2)}(x_1) : x_2 \mapsto W_{(1,2)}(x_1)x_2 := W_{(1,2)}(x_1, x_2)$$
and
\[ W_{[1,2]}(x_2) : x_1 \mapsto W_{[1,2]}(x_2)x_1 := W_{[1,2]}(x_1, x_2) \]

We can prove that when these constraints are not viable under an arbitrary dynamic system of the form

\[
\begin{align*}
  i) & \quad x'_i(t) = c_i(x(t)), \quad i = 1, 2 \\
  ii) & \quad W'_0(t) = \alpha_0(W_0(t)) \\
  iii) & \quad W'_1(t) = \alpha_1(W_1(t)) \\
  iv) & \quad W'_2(t) = \alpha_2(W_2(t)) \\
  v) & \quad W'_{[1,2]}(t) = \alpha_{[1,2]}(W_{[1,2]}(t))
\end{align*}
\]

we can still reestablish viability by involving multipliers \( q \in Y^* \) and correct the above system by the control system

\[
\begin{align*}
  i) & \quad x'_i(t) = c_i(x(t)) - W_1(t)^*q(t) - W_{[1,2]}(t)(x_2(t))^*q(t) \\
  ii) & \quad W'_0(t) = \alpha_0(W_0(t)) - q(t) \\
  iii) & \quad W'_1(t) = \alpha_1(W_1(t)) - x_1(t) \otimes q(t) \\
  iv) & \quad W'_2(t) = \alpha_2(W_2(t)) - x_2(t) \otimes q(t) \\
  v) & \quad W'_{[1,2]}(t) = \alpha_{[1,2]}(W_{[1,2]}(t)) - x_1(t) \otimes x_2(t) \otimes q(t) \\
\end{align*}
\]

where \( q(t) \in N_M(W_{[1,2]}(t)(x_1(t), x_2(t)) + W_1(t)x_1(t) + W_2(t)x_2(t) + W_0(t)) \)

Hence, the structure of this control system involves the transposes \( W'_i(t)q(t) \) and \( W_{[1,2]}(t)(x_j(t))^*q(t) \) \((i = 1, 2)\) in the evolution of the variables \( x_i(t) \), and the tensor products \( x_i(t) \otimes q(t) \) (Hebbian rules) in the evolution of the linear operators \( W_i(t) \), and the tensor product \( x_1(t) \otimes x_2(t) \otimes q(t) \) in the evolution of the bilinear form \( W_{[1,2]} \).

The tensor product \( x_1 \otimes x_2 \otimes q \) is a bilinear operator from \( X_1^* \times X_2^* \) to \( Y^* \) associating with any pair \((p_1, p_2) \in X_1^* \times X_2^* \) the element

\[
(x_1 \otimes x_2 \otimes q)(p_1, p_2) := \langle p_1, x_1 \rangle \langle p_2, x_2 \rangle q
\]

If the vector spaces are supplied with bases, the components of this bilinear form — the “tensors” — can be written

\[
(x_1 \otimes x_2 \otimes q)^j_{i_1, i_2} = x_{1i_1} x_{2i_2} q^j
\]

as the products of the components of the three factors of this tensor product.

Taking \( \alpha_{[1,2]}(W) = 0 \), the evolution of the bi-synaptic tensor \( W_{[1,2]} := (a^j_{i_1, i_2}) \) obeys the differential equation

\[
\frac{d}{dt} a^j_{i_1, i_2}(t) = - x_{1i_1}(t) x_{2i_2}(t) q^j(t)
\]
It states that the velocity of the synaptic tensor is the product of the presynaptic activities of the neurons arriving at the synapse \((i_1, i_2, j)\) and the postsynaptic activity (see [10, Aubin & Burnod]).

We may enrich this problem by introducing memberships \(\chi_i(t) \in [0,1]\) of the two players in a fuzzy coalition \(\chi\) aimed at tuning the action \(x_i(t)\) \((i = 1, 2)\) that we shall regard as the components of a fuzzy coalition. In this framework, the constraint becomes: \(\forall t \geq 0,\)

\[
\chi_1(t)\chi_2(t)W_{\{1,2\}}(t)(x_1(t), x_2(t)) + \chi_1(t)W_1(t)x_1(t) + \chi_2(t)W_2(t)x_2(t) + W_0(t) \in M
\]

If we assume that the evolutions of these \(\chi_i(t)\) are governed by differential equations

\[
\chi_i'(t) = \gamma_i(\chi_i(t)), \quad i = 1, 2
\]

we shall prove that the above constraints are viable under the control system

\[
\begin{align*}
  &i) \quad x_1'(t) = c_1(x(t)) - \chi_1(t)W_1(t)^*q(t) - \chi_1(t)\chi_2(t)W_{\{1,2\}}(t)(x_2(t))^*q(t) \\
  &ii) \quad x_2'(t) = c_2(x(t)) - \chi_2(t)W_2(t)^*q(t) - \chi_1(t)\chi_2(t)W_{\{1,2\}}(t)(x_1(t))^*q(t) \\
  &iii) \quad \chi_1'(t) = \gamma_1(\chi_1(t)) - (q(t), W_1(t)x_1(t) + \chi_2(t)W_{\{1,2\}}(t)(x_1(t), x_2(t))) \\
  &iv) \quad \chi_2'(t) = \gamma_2(\chi_2(t)) - (q(t), W_2(t)x_2(t) + \chi_1(t)W_{\{1,2\}}(t)(x_1(t), x_2(t))) \\
  &v) \quad W_0'(t) = \alpha_0(W_0(t)) - q(t) \\
  &vi) \quad W_1'(t) = \alpha_1(W_1(t)) - \chi_1(t)x_1(t) \otimes q(t) \\
  &vii) \quad W_2'(t) = \alpha_2(W_2(t)) - \chi_2(t)x_2(t) \otimes q(t) \\
  &viii) \quad W_{\{1,2\}}'(t) = \alpha_{\{1,2\}}(W_{\{1,2\}}(t)) - \chi_1(t)\chi_2(t)x_1(t) \otimes x_2(t) \otimes q(t)
  \end{align*}
\]

where \(q(t) \in N_M(\chi_1(t)\chi_2(t)W_{\{1,2\}}(t)(x_1(t), x_2(t)) + \chi_1(t)W_1(t)x_1(t) + \chi_2(t)W_2(t)x_2(t) + W_0(t))\)

Observe that the correction of the velocity \(\chi_1'\) of actor 1’s membership involves the cost of the constraint induced by \(W_{\{1,2\}}\) multiplied by the membership \(\chi_2\) of the other actor, describing an influence proportional to his involvement in the fuzzy coalition, describing a panarguean effect.

We refer to [9, Aubin] for more details on this topic.

6 Viability and Optimality

Interestingly enough, viability theory implies the dynamical programming approach for optimal control.

Denote by \(S(x)\) the set of pairs \((x(\cdot), u(\cdot))\) solutions to the control problem (1) starting from \(x\) at time 0. We consider the minimization problem
\[
V(T,x) = \inf_{(x(t),u(t)) \in S(x)} \inf_{t \in [0,T]} \left( c(T - t, x(t)) + \int_0^t 1(x(\tau), u(\tau)) d\tau \right)
\]

where \( c \) and \( l \) are cost functions. We can prove that the graph of the value function \((T,x) \mapsto V(T,x)\) of this optimal control problem is the capture basin of the graph of the cost function \( c \) under an auxiliary system involving \((f,U)\) and the cost function \( l \). The regulation map of this auxiliary system provides the optimal solutions and the tangential conditions furnish Hamilton-Jacobi-Bellman equations of which the value function is the solution (see for instance ([24, 25, 26, 27, Frankowska] and more recently, [8, Aubin]). This is a very general method covering numerous other dynamic optimization problems.

However, contrary to optimal control theory, viability theory does not require any single decision-maker (or actor, or player) to “guide” the system by optimizing an intertemporal optimality criterion\(^5\).

Furthermore, the choice (even conditional) of the controls is not made once and for all at some initial time, but they can be changed at each instant so as to take into account possible modifications of the environment of the system, allowing therefore for adaptation to viability constraints.

Finally, by not appealing to intertemporal criteria, viability theory does not require any knowledge of the future\(^6\) (even of a stochastic nature.) This is of particular importance when experimentation\(^7\) is not possible or when the phenomenon under study is not periodic. For example, in biological evolution as well as in economics and in the other systems we shall investigate, the dynamics of the system disappear and cannot be recreated.

Hence, forecasting or prediction of the future are not the issues which we shall address in this book.

However, the conclusions of the theorems allow us to reduce the choice of possible evolutions, or to single out impossible future events, or to provide

---

\(^5\)The choice of which is open to question even in static models, even when multicriteria or several decision makers are involved in the model.

\(^6\)Most systems we investigate do involve myopic behavior; while they cannot take into account the future, they are certainly constrained by the past.

\(^7\)Experimentation, by assuming that the evolution of the state of the system starting from a given initial state for a same period of time will be the same whatever the initial time, allows one to translate the time interval back and forth, and, thus, to “know” the future evolution of the system.
explanation of some behaviors which do not fit any reasonable optimality criterion.

Therefore, instead of using intertemporal optimization\(^8\) that involves the future, viability theory provides selection procedures of viable evolutions obeying, at each instant, state constraints which depend upon the present or the past. (This does not exclude anticipations, which are extrapolations of past evolutions, constraining in the last analysis the evolution of the system to be a function of its history.)

References


\(^8\)which can be traced back to Sumerian mythology which is at the origin of Genesis: one Decision-Maker, deciding what is good and bad and choosing the best (fortunately, on an intertemporal basis, thus wisely postponing to eternity the verification of optimality), knowing the future, and having taken the optimal decisions, well, during one week...


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