Statistical Mechanics Approaches in Economics:

Suggested Reading and Interpretations

Gordon M.B., Nadal J.P., Phan D.

Following the presentation by Gordon (this book) of the general principles of statistical mechanics, this paper provides a bird's eye view of literature in economy that exhibits strong links with statistical mechanics. The aim of this literature is to determine aggregate behaviours based on models where individual agents' decisions are subject to social influence or social interaction effects. Durlauf (1997, 2001), Brock, Durlauf (1999, 2000), Blume, Durlauf (2001) (in the following: BBD) treat one aspect of this question, including a review of the literature. Their work is mainly based on a game theoretic approach and has an "extended standard theory" flavour (Faverau, ...). In this paper we extend the review to include some other literature, like interactive decision theory and more heterodox papers. Our aim is to motivate a discussion upon the risk and opportunity of transposing the tools, methods and concepts from statistical mechanics to economics. We use a unified formal framework, close both to Gordon (this book) and Durlauf (2001).

The first two sections are based on a generalized version of standard economic models of individual choices that include the social influence explicitly. First, we review the game theoretic approach, following the framework suggested by BBD. In this perspective, the relevant concept is that of Nash equilibrium. In section 2 we focus on an interactive decision-theoretic approach. In this case, a monopolist faces a customer with unknown personal characteristics, but the global distribution of such characteristics is well known, at least in the absence of social influence (the risky case). In this case, customers have no strategic behaviour. The cognitive problem arises when social influence processes come into play. These reconfigure the global distribution of the willingness to pay, according to effects mediated by the network's structure. The last section is devoted to more heterodox approaches, including the famous Marseille wholesale fish market (Weisbusch-Kirman-Herreiner, 1998; Nadal, Weisbusch, Chenevez, Kirman, 1998)

1. The Game-Theoretic Approach

Brock, Durlauf (1999, 2000)

1.1. Basic Model

In the BBD basic model, one considers a population of N agents. Each agent has to make a binary choice \( \omega_i \) in the set \{-1, 1\}. Agents are assumed to maximise a linear (expected) utility:

\[
\max_{\omega_i \in \{-1,1\}} V_i(\omega_i) = (h_i + \epsilon_i) \omega_i + E_i[S(\omega_i, \omega_{-i})]
\]

This specification embodies both a « private » and a « social » component. The private component includes a deterministic part: \( h_i, \omega_i \), and a stochastic part: \( \epsilon_i, \omega_i \), where the \( \epsilon_i \) are random variables independent of the agent's choice. If the law of \( \epsilon_i \) has zero mean, \( h_i, \omega_i \) can be interpreted as the expected utility of agent \( i \) in the absence of social effects. The social component \( S(\omega_i, \omega_{-i}) \) takes into account the interactive dimension of the decision process, i.e. the social effect on the utility of agent \( i \) due to the behaviour of the other agents. \( \omega_i \) denotes the choices' vector of the neighbours of agent \( i \), a subset of agents denoted \( \vartheta_i \). More specifically, BBD assume information asymmetry: each agent \( i \) knows his own choice, but has to make assumptions about the behaviour of the agents in his « neighbourhood » \( \vartheta_i \). \( E_i[.] \) denotes agent's \( i \) beliefs about the effect of such neighbourhood behaviour on his own utility.

The N equations (1) should be solved simultaneously for all the agents, \( i=1,...,N \). In these equations, classical rationality is relaxed at two different levels. The "noise" \( \epsilon_i \) introduces some indeterminacy in
the private component of the utility. The other source of boundedness in rationality arises when the subjective expectations \( E_i \) are inconsistent with the \textit{a posteriori} probability distribution of choices.

The choice that maximizes (1) is \( \omega = 1 \) if \( h_i + e_i + E_i[S(+1, \omega_{-i})] > -h_i - e_i + E_i[S(-1, \omega_{-i})] \), that is, if \( 2h_i + E_i[S(+1, \omega_{-i})] - E_i[S(-1, \omega_{-i})] > 2e_i \); otherwise \( \omega = -1 \). Thus, the choice depends on the subjective expectations and on the probability law of the random variable \( \epsilon_i \):

\[
P_i(\omega = +1 | E_i[S(+1, \omega_{-i})]; E_i[S(-1, \omega_{-i})]) = P_i\left( \epsilon_i > -h_i + \frac{E_i[S(-1, \omega_{-i})] - E_i[S(+1, \omega_{-i})]}{2} \right)
\]

and \( P_i(\omega = -1 | E_i[S(+1, \omega_{-i})]; E_i[S(-1, \omega_{-i})]) = 1 - P_i(\omega = +1 | E_i[S(+1, \omega_{-i})]; E_i[S(-1, \omega_{-i})]) \).

In the case of rational expectations, the subjective expectations \( E_i \) are the true mathematical expectations \( E[.] \). This imposes a non-trivial of self-consistence condition, further discussed in the section 1.2 (à voir....).

Let us denote \( J_{ik} \) the (marginal) social influence on agent \( i \) due to the decision of agent \( k \in \varnothing_i \). BBD assume a positive influence, i.e. \textit{strategic complementarity} (Bulow, Geanakoplos, Klemperer, 1985, Cooper, John, 1988). Formally, BBD define \( J_{ik} \) as the (positive) second order cross derivative of \( S(\omega_i, \omega_{-i}) \) with respect to \( \omega_i \) and \( \omega_k \) (this definition applies to the restriction to binary arguments \( \omega_i \) and \( \omega_k \) of a continuous function \( S(\omega_i, \omega_{-i}) ):

\[
\frac{\partial^2 S(\omega_i, \omega_{-i})}{\partial \omega_i \partial \omega_k} = J_{ik} > 0
\]

There are at least two simple specifications for \( S \) that satisfy this condition. On one hand, one can assume a \textit{negative quadratic conformity effect} (Berheim, 1994):

\[
S(\omega_i, \omega_{-i}) = -\sum_{k \in \varnothing_i} \frac{J_{ik}}{2}(\omega_i - \omega_k)^2
\]

In this case, when agents are disconnected (\( J_{ik}=0 \)) or when choices are similar (\( \omega_i = \omega_k \)) the social effect vanishes. As soon as the agent’s decision differs from that of one of his neighbours, there is a local effect of negative value. As a consequence, \( 2J_{ik} \) can be interpreted as the loss of agent \( i \) if his own choice \( \omega_i \) is non conformal with the choice of his neighbour \( k \). If \( J_{ik}=J_{ki} \) there is reciprocity.

On the other hand, the social effect can be positive and multiplicative:

\[
S(\omega_i, \omega_{-i}) = \omega_i \sum_{k \in \varnothing_i} J_{ik} \omega_j
\]

Both formulations of the social effects lead to the same optimization problem for the utility (1): since \( \omega_i^2 = \omega_k^2 = 1 \) for all \( i, k \), the quadratic conformity effect (4) may be written as follows:

\[
S(\omega_i, \omega_{-i}) = +\sum_{k \in \varnothing_i} J_{ik} \omega_i \omega_k - \sum_{k \in \varnothing_i} \frac{J_{ik}}{2}(\omega_i^2 + \omega_k^2) = \omega_i \sum_{k \in \varnothing_i} J_{ik} \omega_k - \sum_{k \in \varnothing_i} J_{ik}
\]

As a consequence, formulations (4) and (5) only differ by an irrelevant constant value. Hereafter we use the multiplicative formulation (5). We further assume that each agent \( i \) knows precisely the marginal losses \( J_{ik} \) due to non conformity, and has only to estimate his neighbours’ choices in (5):

\[
E_i[S(\omega_i, \omega_{-i})] = E_i\left[ \omega_i \sum_{k \in \varnothing_i} J_{ik} \omega_k \right] = \omega_i \sum_{k \in \varnothing_i} J_{ik} E_i[\omega_k]
\]

The expectations of agent \( i \) about the social effects is completely determined by his the expectations of his neighbours' choices, and (1) can be rewritten as:

\[
\max_{\omega_i \in [-1,1]} V_i(\omega_i) = \omega_i(h_i + e_i + \sum_{k \in \varnothing_i} J_{ik} E_i[\omega_k])
\]

Introducing these results into (2), we obtain
The analogy with statistical mechanics becomes apparent if one assumes a logistic distribution for the random payoffs $\varepsilon_i$:

$$P(\varepsilon_i \leq z) = \frac{1}{1 + \exp(-2\beta_i z)}$$

where the parameter $\beta_i$ controls the width of the distribution. Notice that a logistic distribution of parameter $\beta$ has the same sigmoidal shape as an error function of variance $\sigma^2 = \frac{\pi^2}{3.6}$. It presents the advantage of having an analytical expression.

With the Logit assumption the probability (8) becomes

$$P_1(\omega_i = +1| \{E_i[\omega_k]\}_{k \in \Omega_i}) = \frac{\exp(+\beta_i z_i)}{\exp(+\beta_i z_i) + \exp(-\beta_i z_i)}$$
$$P_1(\omega_i = -1| \{E_i[\omega_k]\}_{k \in \Omega_i}) = \frac{\exp(-\beta_i z_i)}{\exp(+\beta_i z_i) + \exp(-\beta_i z_i)}$$

with

$$z_i = h_i + \sum_{k \in \Omega_i} J_{ik} E_i[\omega_k].$$

If the individual stochastic terms $\varepsilon_i$ are independent, the joint probability measure over the population follows a Gibbs Law:

$$P(\omega_i, S(\omega_i, \omega_{-i})) = \prod_i P(\omega_i|S(\omega_i, \omega_{-i})) = \prod_i \frac{\exp(\omega_i \beta z_i)}{\exp(-\beta_i z_i) + \exp(+\beta_i z_i)}$$

From (10) we obtain the mathematical expectation of the choice of agent $i$, given his own expectations about the other agents' behaviours:

$$E_i[\omega_i| \{E_i[\omega_k]\}_{k \in \Omega_i}] = \frac{\exp(+\beta_i z_i) - \exp(-\beta_i z_i)}{\exp(+\beta_i z_i) + \exp(-\beta_i z_i)} = \tanh(\beta_i z_i)$$

This equation presents a strong analogy with the expression of the magnetization of an Ising spin in a magnetic field. The analogy is as follows: the total utility $z_i$ is the external field, the expectation $E_i[\omega_i]$ is the average magnetization of the physics literature, and the width of the random payoffs distribution is the inverse of the temperature. If $\beta = \beta_i$ is the same for all agents, (10) is exactly the Gibbs Law for the Ising model in a field $z_i$ at temperature $T = 1/\beta$ presented in Gordon, section 4. When the latter goes to zero, which corresponds to the limit $\beta \to \infty$, the utility is maximized by choices $\omega_i = \text{sign}[z_i]$. If $\beta$ is finite, the probability of choices is given by (10), which leads to the average choice (13). In the limit of very high temperature (small $\beta$) the random payoffs may be so large that both choices have almost the same probability, the corresponding expectations being close to zero. The main difference with Physics is that up to now, the expectations $E_i[\omega_i]$ entering in the definition of $z_i$ have not been specified, whereas in Physics these have to satisfy strong consistency conditions.
1.2. Equilibrium & Dynamics

Until now, no constraints have been imposed on the individual expectations \( \{E_i(\omega_k)\}_{k \in \mathcal{V}_i} \), that at this stage may be arbitrary. Within the special (neo-classical) approach of rational expectations, all the agents have the same, rational, behaviour. This results from the preceding formulation, if one assumes that the expectations of each agent \( i \) about the choices of his neighbours, \( E_i(\omega_k) \) in the right hand side of (8), are consistent with the own expectations of these neighbours, deduced using the probabilities (10). That is, for all \( i \) and all \( k \), rationality imposes:

\[
E_i(\omega_k) = E_k \left[ \omega_k \left[ E_k \left[ \omega_i \right] \right] \right]_{j \in \mathcal{V}_k}
\]

where \( \{E_k[\omega_j]\}_{j \in \mathcal{V}_k} \) stands for the expectations of agent \( k \) about the choices of its own neighbours.

Depending on the kind of neighbourhood, conditions (14) may be more or less easy to satisfy. In the extreme case where the agents' utilities do not depend on social effects, that is \( J_{ik}=0 \) for all \( i,k \), the problem is trivial: each agent maximizes his own utility independently of the others, and the probability of choices is given by (12) with \( z_i = h_i \). Another trivial case is that of complete asymmetry, in which the neighbours of agent \( i \) do not suffer from his social influence; that is, if \( J_{ik}\neq0 \) then \( J_{ki}=0 \).

In the general case we expect that if \( J_{ik}\neq0 \) (the choice of agent \( k \) has an effect on the utility of \( i \), then \( J_{ki}\neq0 \). Clearly, now equations (14) become entangled. A simple example with two agents helps clarifying this point. For, if there is some degree of reciprocity, (14) may be explicitly written as:

\[
\begin{align*}
E_1(\omega_2) &= E_2(\omega_2) E_2(\omega_1) \\
E_2(\omega_1) &= E_1(\omega_1) E_1(\omega_2)
\end{align*}
\]

which is reminiscent of the conditions leading to a Nash equilibrium in game theory. [à vérifier!] These conditions are verified if \( E_1(\omega_2)=E_2(\omega_2) \) and \( E_2(\omega_1)=E_1(\omega_1) \), that is, if the expectation of each agent about the other’s choice is equal to the other’s expectation. Extending this assumption to all the agents, the \( E_i(\omega_k) \) may be replaced by the mathematical expectations \( E_k(\omega_k) \):

\[
E(\omega_k) = E_k(\omega_k)
\]

A further simplification is to assume the same law for the agents’ random payoffs, that is, \( \beta_i=\beta \). In that case, the probability distributions of the agents’ choices follow all the same law, and we may drop the subscript \( k \) in the expectation, to write \( E_k(\omega_k)=E(\omega_k) \). Introducing these assumptions in (11) and (13), we obtain the following system:

\[
E(\omega_k) = \tanh \left( \beta \left( h_i + \sum_{k \in \mathcal{V}_i} J_{ik} E(\omega_k) \right) \right)
\]

Following Brock, Durlauf (1999), system (17) has at least one fixed point.

Consider a system with reciprocity in the social effects. In this case the \( J_{ik} \) are symmetrical, \( J_{ki}=J_{ki} \), and equations (17) are the mean field equations of the Ising model with interactions \( J_{ik} \), in local external fields \( h_i \), at temperature \( \beta^{-1} \). The solution is far from trivial: depending on the sign and range of the interactions \( J_{ik} \), on the distribution of the local fields \( h_i \) and the temperature, many different fixed points may exist. A huge number of fixed points, of the order of \( 2^N \), may exist under some conditions. The model in Physics presenting this behaviour is called a spin-glass.

An interesting case is that of homogeneous local interactions, where all neighbourhoods \( \mathcal{V}_i \) have the same size \( n \), the interaction parameters have all the same value \( J/n \) with \( J>0 \) over the neighbourhood, and the parameter \( \beta_i \) is the same for all the agents. In this case of perfect reciprocity and rationality, all the agents have the same expectations: \( E_i(\omega_k) = E(\omega) \) for all \( k \). The social influence components \( S(\omega_k,\omega_{-i}) \) become:

\[
S(\omega_k,\omega_{-i}) = \omega_i J E(\omega) \quad \forall i
\]

and the stationary social choice is the solution of the single mean field equation of the Ising model:
According with standard results in Statistical Mechanics, there exists either one or three solutions, depending on the magnitudes of the private utility, the width of the stochastic term and the social effects (Brock, Durlauf 2000):

\[
E(\omega) = \tanh[\beta(h + JE(\omega))]
\]

**ICI REPRESENTATION GRAPHIQUE ?**

- i) If \( h=0 \), then
  - a) if \( \beta J < 1 \), then there is a single solution to (19), \( E(\omega)=0 \)
  - b) If \( \beta J > 1 \), there exist three solutions to (19): a positive one \( E(\omega)>0 \), a negative one \( E(\omega)<0 \), and \( E(\omega)=0 \).

- ii) If \( h \neq 0 \), then there is always a solution to (19), such that \( E(\omega) \) has the same sign of \( h \), for all \( \beta \).
  - Besides that,
    - a) if \( \beta J < 1 \), this is the only solution.
    - b) If \( \beta J > 1 \), if \( J>h>0 \) there are two other solutions, with \( E(\omega)<0 \).

Notice that the latter solutions, which only arise if the social effects are strong enough, correspond to average choices that are in contradiction with the private component of the utility function.

**Economic Interpretation** - The magnitude of \( J \) represents the weight of the social influence on the individual choice. In this model, the larger \( J \), the stronger the influence on his own behaviour of the agent's beliefs about the others' choices, and weaker is the relative weight of his own private component, \( h \). For large values of \( J \), these social effects result in the emergence of a specific order, produced by the social interaction. This means that the knowledge of private individual preferences is not sufficient to predict the collective behaviour, as pointed out by Durlauf (2001): "When individual behaviour is driven by a desire to be similar to others, this does not provide any information on what they actually do; rather it merely implies that whatever behaviour occurs, there will be substantial within-group correlation due to the conformity effects". In the case of multiple equilibrium, indeterminacy between solution is resolved by a selection process resulting from collective interdependencies. When the degree of conformism is strong, the processes of interaction can lead the mean social choice to disregard the private part of the surplus function. Such a result is not surprising; it occurs as soon as the agents decisions are the result of compromises between social information and private information, as for instance in the aggregate Fokker-Plank based models of Topol (1991), Orléan (1998a, 1998b, 1998c), or in the Internet chat rooms model of Curien & allii (2001) - see also Curien, Moreau in this issue.